

# Conformal Structures Admitted by a Class of FRW Cosmologies

**Philip Threlfall and Susan M Scott**

Centre for Gravitational Physics,  
College of Physical Sciences,  
The Australian National University,  
Canberra ACT 0200  
Australia

E-mail: [phil.threlfall@anu.edu.au](mailto:phil.threlfall@anu.edu.au), [susan.scott@anu.edu.au](mailto:susan.scott@anu.edu.au)

**Abstract.** In this paper we demonstrate that there are large classes of Friedmann-Robertson-Walker (FRW) cosmologies that admit isotropic conformal structures of Quiescent Cosmology. FRW models have long been known to admit singularities such as Big Bangs and Big Crunches [1, 2] but recently it has been shown that there are other cosmological structures that these solutions contain. These structures are Big Rips, Sudden Singularities and Extremality Events [1, 2]. Within the Quiescent Cosmology framework [3] there also exist structures consistent with a cosmological singularity known as the Isotropic Past Singularity (IPS) [4, 5]. There also exists a cosmological final state known as a Future Isotropic Universe (FIU) [4], which strictly speaking, doesn't fit with the fundamental ideals of Quiescent Cosmology. In this paper, we compare the cosmological events of a large class of FRW solutions to the conformal structures of Quiescent Cosmology [4].

In the first section of this paper we present the relevant background information and our motivation. In the second section of this paper we construct conformal relationships for relevant FRW models. The third section contains a thorough discussion of a class of FRW solutions that cannot represent any of the previously constructed isotropic conformal structures from Quiescent Cosmology. The final section contains our remarks and future outlook for further study of this field.

# 1. Background and Motivation

## 1.1. The FRW Cosmologies

The FRW cosmologies are the simplest, non-trivial solutions to the Einstein field equations that represent homogeneous and isotropic cosmologies which contract or expand [6]. They are of fundamental importance because, although simple, they (at least approximately) describe the physical properties of the observable Universe. The FRW metric is given, in spherical polar coordinates  $(t, r, \theta, \phi)$ , by

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (1)$$

where  $a(t)$  is known as the scale factor and  $k = -1, 0, +1$  represents the curvature of the 3-space.

The most well-known types of cosmological events present in the FRW space-times are the singularities known as the Big Bang if the singularity is in the past and the Big Crunch if the singularity is in the future. In both cases the scale factor goes to zero. Recently there has been considerable interest in a new class of cosmological events, including Big Rips, Sudden Singularities and Extremality Events [1, 2, 7, 8, 9].

## 1.2. Cosmological Events in the FRW Framework

When considering cosmological events it is instructive to consider the scale factor as a generalised Puiseux series expansion around the time,  $t_0$ , when the event occurs, as seen in [1] and [2],

$$a(t) = c_0 |t - t_0|^{\eta_0} + c_1 |t - t_0|^{\eta_1} + c_2 |t - t_0|^{\eta_2} + \dots, \quad (2)$$

where the exponents  $\eta_i \in \mathbb{R}$  are ordered such that  $\eta_i < \eta_{i+1}$ . There are no restrictions on the constants,  $c_i$ , apart from  $c_0 > 0$  and  $c_i \neq 0$ . In the following subsection we will adopt the definitions used in the comprehensive paper by Cattoën and Visser [2] but, in the interests of simplicity, we will suppose that all cosmological events happen at a time  $t_0$ .

*1.2.1. Big Bangs/Big Crunches* A cosmological event is said to be a Big Bang (Big Crunch) if  $a(t) \rightarrow 0$  at some finite time  $t_0$  in the past (future). The scale factors for these singularities will be expressed as

$$a(t) = c_0(t - t_0)^{\eta_0} + c_1(t - t_0)^{\eta_1} + c_2(t - t_0)^{\eta_2} + \dots, \quad (3)$$

for a Big Bang and

$$a(t) = c_0(t_0 - t)^{\eta_0} + c_1(t_0 - t)^{\eta_1} + c_2(t_0 - t)^{\eta_2} + \dots, \quad (4)$$

for a Big Crunch. The exponents will satisfy  $0 < \eta_0 < \eta_1 < \eta_2 < \dots$  and  $\eta_i \in \mathbb{R}^+$ .

*1.2.2. Big Rips* In the opposite cosmological scenario, we can have the scale factor becoming infinite as  $t_0$  is approached to the future. In this case the cosmological event is referred to as a Big Rip. The scale factor has a form similar to that of the Big Crunch but in this case  $\eta_0 < 0$  and  $\eta_0 < \eta_1 < \dots$ :

$$a(t) = c_0(t_0 - t)^{\eta_0} + c_1(t_0 - t)^{\eta_1} + c_2(t_0 - t)^{\eta_2} + \dots, \quad (5)$$

with  $c_0 > 0$  and  $\eta_i \in \mathbb{R}$ .

*1.2.3. Sudden Singularities* We have considered zero and infinite scale factors but we now turn our attention to a finite non-zero scale factor at  $t_0$ . If the scale factor is finite but one of its time derivatives diverges then we will say that the cosmological event is a Sudden Singularity. The scale factor can be written as

$$a(t) = c_0 + c_1|t_0 - t|^{\eta_1} + c_2|t_0 - t|^{\eta_2} + \dots, \quad (6)$$

where we have set  $\eta_0 = 0$  and  $0 < \eta_1 < \eta_2 < \dots$ ,  $\eta_i \in \mathbb{R}^+$ . We also require, for some  $i$ , with  $\eta_i \notin \mathbb{N}$ , that a time derivative,

$$a^n(t) \sim c_i \eta_i (\eta_i - 1) \dots (\eta_i - n + 1) |t_0 - t|^{\eta_i - n}, \quad (7)$$

diverges as  $t \rightarrow t_0$ .

*1.2.4. Extremality Events* This class of events does not contain any singularities. For Extremality Events the scale factor  $a(t)$  exhibits a local extremum at finite time  $t_0$ ; in particular  $a'(t) \rightarrow 0$  as  $t \rightarrow t_0$ . It has the same form as the scale factor for Sudden Singularities but  $\eta_i \in \mathbb{N}$  for all  $i > 0$ .

- A ‘bounce’ is any local minimum of  $a(t)$  such that  $a'(t_0) = 0$ . The ‘order’ of the bounce is the first non-zero integer  $n$  for which

$$a^{(2n)}(t_0) > 0, \quad (8)$$

so that

$$a(t) = a(t_0) + \frac{1}{(2n)!} a^{(2n)}(t_0) [t_0 - t]^{2n} + \dots \quad (9)$$

- A ‘turnaround’ is any local maximum of  $a(t)$  such that  $a'(t_0) = 0$ . The ‘order’ of the turnaround is the first non-zero integer  $n$  for which

$$a^{(2n)}(t_0) < 0, \quad (10)$$

so that

$$a(t) = a(t_0) + \frac{1}{(2n)!} a^{(2n)}(t_0) [t_0 - t]^{2n} + \dots \quad (11)$$

- An ‘inflexion event’ is an Extremality Event that is neither a local minimum nor a local maximum. The ‘order’ of an inflexion event is the first non-zero integer  $n$  for which

$$a^{(2n+1)}(t_0) \neq 0, \quad (12)$$

so that

$$a(t) = a(t_0) + \frac{1}{(2n+1)!} a^{(2n+1)}(t_0) [t_0 - t]^{2n+1} + \dots \quad (13)$$

### 1.3. Quiescent Cosmology

According to classical General Relativity, it is generally agreed that the Universe we observe today started with a Big Bang, there remains some debate as to the order of the primordial quantum foam from which the Universe was forged. It can be argued that the Universe was highly ordered in the beginning or that it was highly disordered. The former leads to the formulation of Quiescent Cosmology [3], the latter to Chaotic Cosmology [10]. We shall concern ourselves here with Quiescent Cosmology and consider the associated conformal relationships of the metric.

Quiescent Cosmology was introduced by Barrow in 1978 [3] and effectively states that the Universe began in a highly ordered state and has evolved away from its highly

regular and smooth beginning because of gravitational attraction. We continue to see isotropy on large scales because we exist in an early stage of cosmological evolution. The kind of initial cosmological singularity that is compatible with the Quiescent Cosmology concept is one that is isotropic. This type of initial isotropic singularity was given a rigorous mathematical definition in 1985 by Goode and Wainwright [5] using a conformal relationship of the metric. The version of the definition given below is due to Scott [11] who removed the inherent technical redundancies of the original definition.

Goode and Wainwright were concerned with the past of the Universe (Big Bang). In recent years, however, there has been increasing interest in possible future evolutions of the Universe and its eventual fate. In 2009 Höhn and Scott [4] investigated possible isotropic and anisotropic future end states of the Universe and introduced new definitions to cover the various possible scenarios. Following Goode and Wainwright they also employed conformal relationships of the metric.

#### *1.4. Conformal Structures*

Since, in this paper, we are dealing with FRW space-times which are homogeneous and isotropic, we will present here only the conformal definitions that relate to isotropic initial and final states of the Universe. These will comprise the Isotropic Past Singularity, the Isotropic Future Singularity and the Future Isotropic Universe. Any ancillary definitions that are needed will also be included.

**Definition 1** (Conformally related metric) A metric  $\mathbf{g}$  is said to be conformally related to a metric  $\tilde{\mathbf{g}}$  on a manifold  $\mathcal{M}$  if there exists a conformal factor  $\Omega$  such that

$$\mathbf{g} = \Omega^2 \tilde{\mathbf{g}}, \text{ where } \Omega \text{ is a strictly positive function on } \mathcal{M}. \quad (14)$$

**Definition 2** (Cosmic time function) For a space-time  $(\mathcal{M}, g)$ , a cosmic time function is a function  $T$  on the manifold  $\mathcal{M}$  which increases along every future-directed causal curve.

It should be noted that we will henceforth denote relevant quantities for past

cosmological frameworks with a tilde ( $\sim$ ) and for future cosmological frameworks we will use a bar ( $-$ ).

**Definition 3** (Isotropic Past Singularity (IPS)) A space-time  $(\mathcal{M}, \mathbf{g})$  admits an Isotropic Past Singularity if there exists a space-time  $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ , a smooth cosmic time function  $T$  defined on  $\tilde{\mathcal{M}}$  and a conformal factor  $\Omega(T)$  which satisfy

- i)  $\mathcal{M}$  is the open submanifold  $T > 0$ ,
- ii)  $\mathbf{g} = \Omega^2(T)\tilde{\mathbf{g}}$  on  $\mathcal{M}$ , with  $\tilde{\mathbf{g}}$  regular (at least  $C^3$  and non-degenerate) on an open neighbourhood of  $T = 0$ ,
- iii)  $\Omega(0) = 0$  and  $\exists b > 0$  such that  $\Omega \in C^0[0, b] \cap C^3(0, b]$  and  $\Omega'(0, b] > 0$ ,
- iv)  $\lambda \equiv \lim_{T \rightarrow 0^+} L(T)$  exists,  $\lambda \neq 1$ , where  $L \equiv \frac{\Omega''}{\Omega} \left(\frac{\Omega}{\Omega'}\right)^2$  and a prime denotes differentiation with respect to  $T$ .

It was demonstrated by Goode and Wainwright [5] that, in order to ensure initial asymptotic isotropy, it is also necessary to introduce a constraint on the cosmological fluid flow.

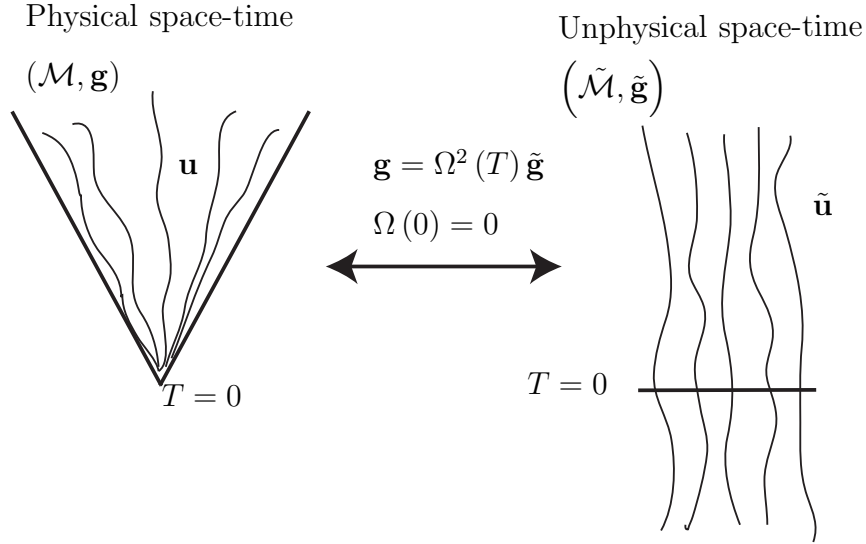
**Definition 4** (IPS fluid congruence) With any unit timelike congruence  $\mathbf{u}$  in  $\mathcal{M}$  we can associate a unit timelike congruence  $\tilde{\mathbf{u}}$  in  $\tilde{\mathcal{M}}$  such that

$$\tilde{\mathbf{u}} = \Omega \mathbf{u} \quad \text{in } \mathcal{M}. \quad (15)$$

- a) If we can choose  $\tilde{\mathbf{u}}$  to be regular (at least  $C^3$ ) on an open neighbourhood of  $T = 0$  in  $\tilde{\mathcal{M}}$ , we say that  $\mathbf{u}$  is regular at the IPS.
- b) If, in addition,  $\tilde{\mathbf{u}}$  is orthogonal to  $T = 0$ , we say that  $\mathbf{u}$  is orthogonal to the IPS.

In figure 1 we present a pictorial interpretation of the IPS.

**Figure 1.** A pictorial interpretation of an IPS. The fluid flow is represented by  $\mathbf{u}$ .



Below is given the analogous definition of an Isotropic Future Singularity introduced by Höhn and Scott [4], followed by the constraint on the fluid flow required to ensure final asymptotic isotropy.

**Definition 5** (Isotropic Future Singularity (IFS)) A space-time  $(\mathcal{M}, \mathbf{g})$  admits an Isotropic Future Singularity if there exists a space-time  $(\bar{\mathcal{M}}, \bar{\mathbf{g}})$ , a smooth cosmic time function  $\bar{T}$  defined on  $\bar{\mathcal{M}}$ , and a conformal factor  $\bar{\Omega}(\bar{T})$  which satisfy

- i)  $\mathcal{M}$  is the open submanifold  $\bar{T} < 0$ ,
- ii)  $\mathbf{g} = \bar{\Omega}^2(\bar{T})\bar{\mathbf{g}}$  on  $\mathcal{M}$ , with  $\bar{\mathbf{g}}$  regular (at least  $C^2$  and non-degenerate) on an open neighbourhood of  $\bar{T} = 0$ ,
- iii)  $\bar{\Omega}(0) = 0$  and  $\exists c > 0$  such that  $\bar{\Omega} \in C^0[-c, 0] \cap C^2[-c, 0)$  and  $\bar{\Omega}$  is positive on  $[-c, 0)$ ,
- iv)  $\bar{\lambda} \equiv \lim_{\bar{T} \rightarrow 0^-} \bar{L}(\bar{T})$  exists,  $\bar{\lambda} \neq 1$ , where  $\bar{L} \equiv \frac{\bar{\Omega}''}{\bar{\Omega}} \left( \frac{\bar{\Omega}}{\bar{\Omega}'} \right)^2$  and a prime denotes differentiation with respect to  $\bar{T}$ .

**Definition 6** (IFS fluid congruence) With any unit timelike congruence  $\mathbf{u}$  in  $\mathcal{M}$  we

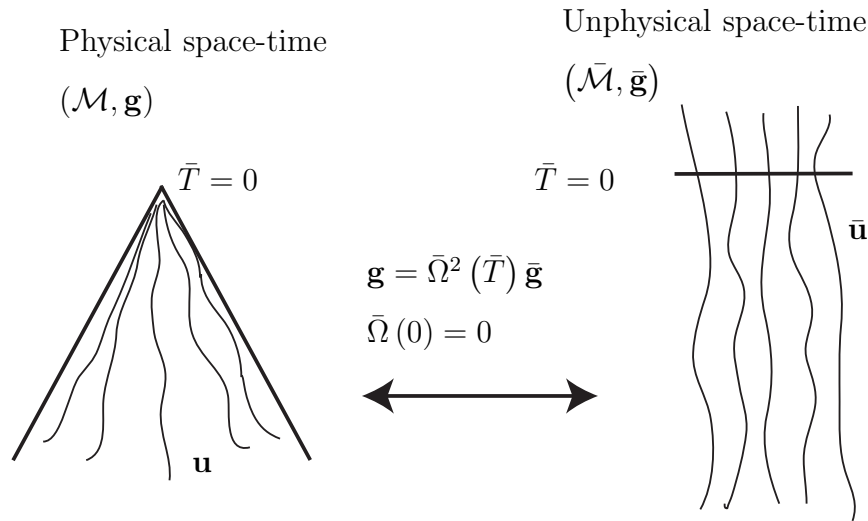
can associate a unit timelike congruence  $\bar{\mathbf{u}}$  in  $\bar{\mathcal{M}}$  such that

$$\bar{\mathbf{u}} = \bar{\Omega} \mathbf{u} \quad \text{in } \mathcal{M}. \quad (16)$$

- a) If we can choose  $\bar{\mathbf{u}}$  to be regular (at least  $C^2$ ) on an open neighbourhood of  $\bar{T} = 0$  in  $\bar{\mathcal{M}}$ , we say that  $\mathbf{u}$  is regular at the IFS.
- b) If, in addition,  $\bar{\mathbf{u}}$  is orthogonal to  $\bar{T} = 0$ , we say that  $\mathbf{u}$  is orthogonal to the IFS.

In figure 2 we present a pictorial interpretation of the IFS.

**Figure 2.** A pictorial interpretation of an IFS. The fluid flow is represented by  $\mathbf{u}$ . It can be seen that an IFS is essentially a time reversal of an IPS.



Finally we give below the definition of a Future Isotropic Universe introduced by Höhn and Scott [4]. This definition covers the further possibility for a conformal structure with an isotropic future behaviour, which does not necessarily lead to a future singularity; for example, some open FRW universes satisfy this definition (see section 2).

**Definition 7** (Future Isotropic Universe (FIU)) A space-time  $(\mathcal{M}, \mathbf{g})$  is said to be a Future Isotropic Universe if there exists a space-time  $(\bar{\mathcal{M}}, \bar{\mathbf{g}})$ , a smooth cosmic time function  $\bar{T}$  defined on  $\bar{\mathcal{M}}$ , and a conformal factor  $\bar{\Omega}(\bar{T})$  which satisfy



- i)  $\lim_{\bar{T} \rightarrow 0^-} \bar{\Omega}(\bar{T}) = +\infty$  and  $\exists c > 0$  such that  $\bar{\Omega} \in C^2[-c, 0)$  and  $\bar{\Omega}$  is strictly monotonically increasing and positive on  $[-c, 0)$ ,
- ii)  $\bar{\lambda}$  as defined above exists,  $\bar{\lambda} \neq 1, 2$ , and  $\bar{L}$  is continuous on  $[-c, 0)$  and
- iii) otherwise the conditions of definitions 5 and 6 are satisfied.

## 2. Conformal Structures for the Cases $\eta_0 < 0$ and $0 < \eta_0 < 1$

We recall the expression, given in section 1.2, for the scale factor as a generalised Puiseux series expansion around the time,  $t_0$ , when a cosmological event occurs:

$$a(t) = c_0|t - t_0|^{\eta_0} + c_1|t - t_0|^{\eta_1} + c_2|t - t_0|^{\eta_2} + \dots \quad c_0 > 0. \quad (17)$$

Define  $\tau = t - t_0$ , so that  $\tau \rightarrow 0^+$  at an initial singularity and  $\tau \rightarrow 0^-$  at a final cosmological event. This means that the scale factor can be expressed as

$$a(\tau) = c_0|\tau|^{\eta_0} + c_1|\tau|^{\eta_1} + c_2|\tau|^{\eta_2} + \dots \quad (18)$$

$$= c_0|\tau|^{\eta_0} \left( 1 + \frac{c_1}{c_0}|\tau|^{\eta_1 - \eta_0} + \frac{c_2}{c_0}|\tau|^{\eta_2 - \eta_0} + \dots \right), \quad \eta_1 - \eta_0 > 0 \quad (19)$$

The metric of the FRW space-time  $(\mathcal{M}, g)$  is given, in the coordinates  $(\tau, r, \theta, \phi)$ , by

$$ds^2 = -d\tau^2 + a^2(\tau) \left( \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right) \quad (20)$$

$$= -d\tau^2 + c_0^2 \tau^{2\eta_0} \left( 1 + 2\frac{c_1}{c_0}|\tau|^{\eta_1 - \eta_0} + \dots \right) d\sigma^2, \quad (21)$$

where  $d\sigma^2 = \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$ .

Apart from the standard coordinate singularities, due to the use of spherical polar coordinates, the metric has the following behaviour:

- (i) if  $\eta_0 < 0$  then there is a metric singularity at  $\tau = 0$ , due to certain metric components becoming infinite,
- (ii) if  $\eta_0 > 0$  then the metric becomes degenerate at  $\tau = 0$ ,
- (iii) if  $\eta_0 = 0$  the metric may be regular at  $\tau = 0$  or singular if any of the time derivatives with respect to  $\tau$  diverge at  $\tau = 0$ .

In light of these facts, and using the previous section as motivation, we will now investigate how these three cases can be accommodated using the conformal frameworks introduced by Höhn and Scott.

### 2.1. The Case $\eta_0 < 0$

For an initial cosmological event define a new time coordinate  $T$  by  $T = \tau^\alpha$ ,  $\alpha > 0$ . So  $T \rightarrow 0^+$  as  $\tau \rightarrow 0^+$ . Define the manifold  $\tilde{\mathcal{M}}$  to have the coordinate patch  $(T, r, \theta, \phi)$ , so that  $\mathcal{M}$  is the open submanifold  $T > 0$  of  $\tilde{\mathcal{M}}$ .

For a final cosmological event define another new time coordinate  $\bar{T}$  by  $(-\bar{T}) = (-\tau)^\alpha$ ,  $\alpha > 0$ . So  $\bar{T} \rightarrow 0^-$  as  $\tau \rightarrow 0^-$ . Now define the manifold  $\bar{\mathcal{M}}$  to have the coordinate patch  $(\bar{T}, r, \theta, \phi)$ , so that  $\mathcal{M}$  is the open submanifold  $\bar{T} < 0$  of  $\bar{\mathcal{M}}$ .

The metrics for the initial cosmological event and the final cosmological event can now be expressed as, respectively,

$$ds^2 = -\frac{1}{\alpha^2} T^{\frac{2}{\alpha}-2} dT^2 + c_0^2 T^{\frac{2\eta_0}{\alpha}} \left( 1 + 2\frac{c_1}{c_0} T^{\frac{\eta_1-\eta_0}{\alpha}} + \dots \right) d\sigma^2 \quad (22)$$

and

$$ds^2 = -\frac{1}{\alpha^2} (-\bar{T})^{\frac{2}{\alpha}-2} d\bar{T}^2 + c_0^2 (-\bar{T})^{\frac{2\eta_0}{\alpha}} \left( 1 + 2\frac{c_1}{c_0} (-\bar{T})^{\frac{\eta_1-\eta_0}{\alpha}} + \dots \right) d\sigma^2. \quad (23)$$

If we set  $\alpha = 1 - \eta_0$  (so that the range of  $\alpha$  is  $\alpha > 1$ ) then the following conformal factors can be extracted, respectively, from the metrics:

$$\Omega(T) = T^{\frac{\eta_0}{1-\eta_0}}, \quad (24)$$

$$\bar{\Omega}(\bar{T}) = (-\bar{T})^{\frac{\eta_0}{1-\eta_0}}. \quad (25)$$

Since  $\eta_0 < 0$  the exponent in each of these conformal factors will be restricted to the range

$$-1 < \frac{\eta_0}{1-\eta_0} < 0. \quad (26)$$

Using this, it can be seen that

- (i)  $\lim_{T \rightarrow 0^+} \Omega(T) = +\infty$  and  $\exists c > 0$  such that  $\Omega \in C^2(0, c]$  and  $\Omega$  is strictly monotonically decreasing and positive on  $(0, c]$ ,

- (ii)  $\lim_{\bar{T} \rightarrow 0^-} \bar{\Omega}(\bar{T}) = +\infty$  and  $\exists c > 0$  such that  $\bar{\Omega} \in C^2[-c, 0)$  and  $\bar{\Omega}$  is strictly monotonically increasing and positive on  $[-c, 0)$ .

We now list the first two derivatives of the conformal factors as well as  $L$  and  $\bar{L}$ :

$$\begin{aligned} \Omega' &= \frac{\eta_0}{1 - \eta_0} T^{\frac{2\eta_0 - 1}{1 - \eta_0}} & \bar{\Omega}' &= -\frac{\eta_0}{1 - \eta_0} (-\bar{T})^{\frac{2\eta_0 - 1}{1 - \eta_0}} \\ \Omega'' &= \frac{\eta_0(2\eta_0 - 1)}{(1 - \eta_0)^2} T^{\frac{3\eta_0 - 2}{1 - \eta_0}} & \bar{\Omega}'' &= \frac{\eta_0(2\eta_0 - 1)}{(1 - \eta_0)^2} (-\bar{T})^{\frac{3\eta_0 - 2}{1 - \eta_0}} \\ L &= \frac{\Omega''\Omega}{\Omega'^2} = 2 - \frac{1}{\eta_0} & \bar{L} &= \frac{\bar{\Omega}''\bar{\Omega}}{\bar{\Omega}'^2} = 2 - \frac{1}{\eta_0}. \end{aligned}$$

We note that  $L$  and  $\bar{L}$  are constant functions and so are continuous on  $(0, c]$  and  $[-c, 0)$  respectively. Also  $\lambda \equiv \lim_{T \rightarrow 0^+} L(T) = 2 - \frac{1}{\eta_0}$  and  $\bar{\lambda} \equiv \lim_{\bar{T} \rightarrow 0^-} \bar{L}(\bar{T}) = 2 - \frac{1}{\eta_0}$  and since  $2 < 2 - \frac{1}{\eta_0}$  it is clear that  $\lambda, \bar{\lambda} \neq 1, 2$ .

The conformally related unphysical metrics,  $\tilde{g}$  and  $\bar{g}$ , are given by

$$d\tilde{s}^2 = -\frac{1}{(1 - \eta_0)^2} dT^2 + c_0^2 \left( 1 + 2\frac{c_1}{c_0} T^{\frac{\eta_1 - \eta_0}{1 - \eta_0}} + \dots \right) d\sigma^2, \quad (27)$$

$$d\bar{s}^2 = -\frac{1}{(1 - \eta_0)^2} d\bar{T}^2 + c_0^2 \left( 1 + 2\frac{c_1}{c_0} (-\bar{T})^{\frac{\eta_1 - \eta_0}{1 - \eta_0}} + \dots \right) d\sigma^2, \quad (28)$$

from which it is clear that the coordinates  $T$  and  $\bar{T}$  are smooth cosmic time functions on  $\tilde{\mathcal{M}}$  and  $\bar{\mathcal{M}}$  respectively. Since  $\frac{\eta_1 - \eta_0}{1 - \eta_0} > 0$ , the metrics  $\tilde{g}$  and  $\bar{g}$  are at least  $C^0$  and non-degenerate on open neighbourhoods of  $T = 0$  and  $\bar{T} = 0$  respectively. The precise differentiability of the metrics fall into three classes:

- (i) If  $\eta_1 \geq 2 - \eta_0 > 2$ , then  $\tilde{g}$  and  $\bar{g}$  are at least  $C^2$
- (ii) If  $1 \leq \eta_1 < 2 - \eta_0$ , then  $\tilde{g}$  and  $\bar{g}$  are  $C^1$  (for  $\eta_1 = 1$ , at least  $C^1$ )
- (iii) If  $\eta_0 < \eta_1 < 1$ , then  $\tilde{g}$  and  $\bar{g}$  are  $C^0$ .

The cosmological fluid flow  $u$  given by

$$u = (1 - \eta_0) T^{-\frac{\eta_0}{1 - \eta_0}} \frac{\partial}{\partial T} \quad \text{for an initial cosmological event} \quad (29)$$

$$u = (1 - \eta_0) (-\bar{T})^{-\frac{\eta_0}{1 - \eta_0}} \frac{\partial}{\partial \bar{T}} \quad \text{for a final cosmological event} \quad (30)$$

is a unit timelike congruence in  $\mathcal{M}$ . The corresponding unit timelike congruences  $\tilde{u}$  in  $\tilde{\mathcal{M}}$  and  $\bar{u}$  in  $\bar{\mathcal{M}}$  are

$$\tilde{u} = \Omega u = (1 - \eta_0) \frac{\partial}{\partial T} \quad (31)$$

$$\bar{u} = \bar{\Omega}u = (1 - \eta_0) \frac{\partial}{\partial \bar{T}}. \quad (32)$$

It is clear that we can choose  $\tilde{u}$  and  $\bar{u}$  to be regular (at least  $C^2$ ) on an open neighbourhood of  $T = 0$  in  $\tilde{\mathcal{M}}$  and an open neighbourhood of  $\bar{T} = 0$  in  $\bar{\mathcal{M}}$  respectively. Additionally  $\tilde{u}$  and  $\bar{u}$  are orthogonal to  $T = 0$  and  $\bar{T} = 0$  respectively.

We have examined the FRW space-times for a general expansion of the scale factor for the case  $\eta_0 < 0$  and have seen that these metrics all exhibit metric singularities. The scale factor has the behaviour  $a(t) \rightarrow +\infty$  as  $t \rightarrow t_0$  so these space-times all possess Big Rips [2] in either their past or future. In the conformal framework given by Höhn and Scott the physical metrics represent a Past Isotropic Universe or a Future Isotropic Universe. At the cosmological event in question, they are conformally related to a regular metric which exists on a larger manifold; these conformally related regular metrics have varying degrees of differentiability. The conformal factor diverges at the cosmological event and the unphysical fluid flow is regular on an open neighbourhood of the event and orthogonal to it.

## 2.2. The Case $0 < \eta_0 < 1$

Having examined the case  $\eta_0 < 0$  in detail we now turn our attention to the case  $\eta_0 > 0$ . The investigation of the case  $0 < \eta_0 < 1$  is very similar to the case  $\eta_0 < 0$  and is presented first. We only present the analysis for a final cosmological event; the initial cosmological event follows logically as seen in the case  $\eta_0 < 0$ .

Define a new time coordinate  $\bar{T}$  by  $(-\bar{T}) = (-\tau)^\alpha$ ,  $\alpha > 0$ . So  $\bar{T} \rightarrow 0^-$  as  $\tau \rightarrow 0^-$ . Now define the manifold  $\bar{\mathcal{M}}$  to have the coordinate patch  $(\bar{T}, r, \theta, \phi)$  so that  $\mathcal{M}$  is the open submanifold  $\bar{T} < 0$  of  $\bar{\mathcal{M}}$ .

The metric for the final cosmological event can be reexpressed as

$$ds^2 = -\frac{1}{\alpha^2}(-\bar{T})^{\frac{2}{\alpha}-2}d\bar{T}^2 + c_0^2(-\bar{T})^{\frac{2\eta_0}{\alpha}} \left( 1 + 2\frac{c_1}{c_0}(-\bar{T})^{\frac{\eta_1-\eta_0}{\alpha}} + \dots \right) d\sigma^2 \quad (33)$$

If we set  $\alpha = 1 - \eta_0$  (so that the range of  $\alpha$  is  $0 < \alpha < 1$ ) then the following conformal

factor can be extracted from the metric:

$$\bar{\Omega}(\bar{T}) = (-\bar{T})^{\frac{\eta_0}{1-\eta_0}}. \quad (34)$$

Since  $0 < \eta_0 < 1$  the exponent will always be greater than zero. Using this, it can be seen that

- (i)  $\lim_{\bar{T} \rightarrow 0^-} \bar{\Omega}(\bar{T}) = 0$  and  $\exists c > 0$  such that  $\bar{\Omega} \in C^0[-c, 0] \cap C^2[-c, 0)$  and  $\bar{\Omega}$  is strictly monotonically decreasing and positive on  $[-c, 0)$ .

We note that  $\bar{L}$  is the same as for the PIU/FIU seen previously

$$\bar{L} = \frac{\bar{\Omega}''\bar{\Omega}}{\bar{\Omega}'^2} = 2 - \frac{1}{\eta_0}.$$

Also  $\bar{\lambda} \equiv \lim_{\bar{T} \rightarrow 0^-} \bar{L}(\bar{T}) = 2 - \frac{1}{\eta_0} < 1$ . The conformally related metric  $\bar{g}$  is the same as the metric for the FIU

$$d\bar{s}^2 = -\frac{1}{(1-\eta_0)^2} d\bar{T}^2 + c_0^2 \left( 1 + 2\frac{c_1}{c_0} (-\bar{T})^{\frac{\eta_1-\eta_0}{1-\eta_0}} + \dots \right) d\sigma^2, \quad (35)$$

from which it is clear that the coordinate  $\bar{T}$  is a smooth cosmic time function on  $\bar{\mathcal{M}}$ . Since  $\frac{\eta_1-\eta_0}{1-\eta_0} > 0$ , the metric  $\bar{g}$  is at least  $C^0$  and non-degenerate on an open neighbourhood of  $\bar{T} = 0$ . The precise differentiability of the metric again falls into three classes:

- (i) If  $\eta_1 \geq 2 - \eta_0 \in (1, 2)$ , then  $\bar{g}$  is at least  $C^2$
- (ii) If  $1 \leq \eta_1 < 2 - \eta_0 \in (1, 2)$ , then  $\bar{g}$  is  $C^1$  (for  $\eta_1 = 1$ , at least  $C^1$ )
- (iii) If  $\eta_0 < \eta_1 < 1$ , then  $\bar{g}$  is  $C^0$ .

The cosmological fluid flow  $u$  given by

$$u = (1 - \eta_0)(-\bar{T})^{-\frac{\eta_0}{1-\eta_0}} \frac{\partial}{\partial \bar{T}} \quad (36)$$

is a unit timelike congruence in  $\mathcal{M}$ . The corresponding unit timelike congruence  $\bar{u}$  in  $\bar{\mathcal{M}}$  is

$$\bar{u} = \bar{\Omega}u = (1 - \eta_0) \frac{\partial}{\partial \bar{T}}. \quad (37)$$

Again, it is clear that we can choose  $\bar{u}$  to be regular (at least  $C^2$ ) on an open neighbourhood of  $\bar{T} = 0$  in  $\bar{\mathcal{M}}$  and that  $\bar{u}$  is orthogonal to  $\bar{T} = 0$ .

We have examined the case  $\eta_0 \in (0, 1)$  and have seen that the metric exhibits a metric singularity. The scale factor has the behaviour  $a(t) \rightarrow 0$  as  $t \rightarrow t_0$  so the metric possesses a Big Bang if the singularity is in the past or a Big Crunch if the singularity is in the future [2]. According to Höhn and Scott this metric represents an Isotropic Past Singularity or an Isotropic Future Singularity respectively. At the cosmological event in question, the metric is conformally related to a regular metric which exists on a larger manifold; the conformally related regular metric has varying degrees of differentiability. The conformal factor converges to 0 at the cosmological event and the unphysical fluid flow is regular on an open neighbourhood of the event and orthogonal to it.

### 3. The Case $\eta_0 \geq 1$

Logically, the next case we would want to consider is for  $\eta_0 \geq 1$ . This case is substantially more involved than the cases  $\eta_0 < 0$  and  $0 < \eta_0 < 1$  and requires a great deal of care.

The first thing we demonstrate is that these solutions cannot represent an IPS/IFS. We need to see, for the case of  $\eta_0 \geq 1$ , if  $\frac{\ddot{a}a}{(\dot{a})^2} \geq 0$ . If this is true then this class of FRW models cannot admit an IPS/IFS (due to The FRW Result of Ericksson [12]).

The scale factor and its derivatives are

$$a(\tau) = \sum_{i=0} c_i \tau^{\eta_i} \quad (38)$$

$$\dot{a}(\tau) = \sum_{i=0} \eta_i c_i \tau^{\eta_i-1} \quad (39)$$

$$\ddot{a}(\tau) = \sum_{i=0} \eta_i (\eta_i - 1) c_i \tau^{\eta_i-2} \quad (40)$$

where  $c_0 \neq 0$ ,  $\eta_0 \neq 0$  and  $\eta_0 \neq 1$ .

To leading order we see,

$$\frac{\ddot{a}(\tau)a(\tau)}{\dot{a}^2(\tau)} \approx \frac{c_0 \eta_0 (\eta_0 - 1) \tau^{\eta_0-2} \cdot c_0 \tau^{\eta_0}}{(c_0 \eta_0 \tau^{\eta_0-1})^2} \quad (41)$$

$$= \frac{\eta_0 - 1}{\eta_0}. \quad (42)$$

So we see that

$$\eta_0 < 0 \implies \frac{\ddot{a}a}{\dot{a}^2} > 1 \quad (43)$$

$$0 < \eta_0 < 1 \implies \frac{\ddot{a}a}{\dot{a}^2} < 0 \quad (44)$$

$$\eta_0 = 1 \implies \lim_{\tau \rightarrow 0^\pm} \frac{\ddot{a}a}{\dot{a}^2} \rightarrow 0 \quad (45)$$

$$\eta_0 > 1 \implies 0 < \frac{\ddot{a}a}{\dot{a}^2} < 1. \quad (46)$$

This demonstrates that, for  $\eta_0 \geq 1$  and for  $\eta_0 < 0$ , the function  $\frac{\ddot{a}a}{\dot{a}^2} > 0$  and as such these type of FRW models cannot admit an IPS/IFS [12] at which the fluid flow is regular. We do know that for  $\eta_0 < 0$ , these FRW models obey the PIU/FIU definition as seen in section 2.1.

We seek to exclude the possibility that for  $\eta_0 \geq 1$ , these solutions admit a PIU/FIU; this is the direction of our analysis now. This study is very similar to the FRW result of Ericksson [12]. We will need to prove a number of preliminary results before we present that result and provide those first.

We introduce the following condition.

**Condition 8** The spacetime  $(\mathcal{M}, g)$  is a  $C^3$  solution of the Einstein Field Equations with perfect fluid source and the unit timelike fluid congruence  $u$  is regular at a PIU/FIU.

This condition is used to consider perfect fluid sources (the FRW solutions fall within this).

**Theorem 9** If the spacetime  $(\mathcal{M}, g)$  satisfies condition 8 then the unit timelike fluid congruence  $\bar{u}$  is orthogonal to the PIU/FIU.

*Proof*

Similar proofs have previously been presented first by Scott [11] and subsequently Ericksson [12].

For a perfect fluid the Einstein Field Equations imply that the rank one tensor representing the anisotropic parts of the Ricci tensor,  $\Sigma_a$ , vanishes. Using this as motivation, we proceed as follows

$$\Sigma_a = 0 \iff \Sigma^a = 0 \quad (47)$$

$$\implies \Sigma^a \bar{T}_{,a} = 0 \quad (48)$$

where

$$\Sigma^a = \bar{\Omega}^{-3} \left( \bar{\Sigma}^a + 2\bar{h}^{ab} (\bar{\Omega}^{-1} \bar{\Omega}_{;bc} - 2\bar{\Omega}^{-2} \bar{\Omega}_{,b} \bar{\Omega}_{,c}) \bar{u}^c \right). \quad (49)$$

This means that we can write the equation  $\Sigma^a \bar{T}_{,a} = 0$  as

$$\bar{\Omega}^{-3} \left( \bar{\Sigma}^a \bar{T}_{,a} + 2\bar{h}^{ab} \bar{T}_{,a} (\bar{\Omega}^{-1} \bar{\Omega}_{;bc} - 2\bar{\Omega}^{-2} \bar{\Omega}_{,b} \bar{\Omega}_{,c}) \bar{u}^c \right) = 0. \quad (50)$$

If the conformal factor is a function of cosmic time only,  $\bar{\Omega} = \bar{\Omega}(\bar{T})$ , as is true for our analysis, then

$$\bar{\Omega}_{,a} = \bar{\Omega}' \bar{T}_{,a} \quad (51)$$

$$\bar{\Omega}_{;ab} = \bar{\Omega}' \left( \bar{T}_{;ab} + \frac{\bar{\Omega}''}{\bar{\Omega}'} \bar{T}_{,a} \bar{T}_{,b} \right). \quad (52)$$

This simplifies equation 50 to

$$\bar{\Omega}^{-3} \left( \bar{\Sigma}^a \bar{T}_{,a} + 2\bar{h}^{ab} \bar{T}_{,a} \frac{\bar{\Omega}'}{\bar{\Omega}} \left( \bar{T}_{;bc} + (\bar{L} - 2) \frac{\bar{\Omega}'}{\bar{\Omega}} \bar{T}_{,b} \bar{T}_{,c} \right) \bar{u}^c \right) = 0 \quad (53)$$

From the definition of a PIU/FIU we know that  $\exists b > 0$  such that  $\bar{\Omega}$  is positive on  $[-b, 0)$ , also  $\exists c \in (0, b]$  such that  $\bar{\Omega}' \neq 0$  on  $[-c, 0)$ . So therefore

$$\left( \frac{\bar{\Omega}}{\bar{\Omega}'} \right)^2 \bar{\Sigma}^a \bar{T}_{,a} + 2 \frac{\bar{\Omega}}{\bar{\Omega}'} \bar{h}^{ab} \bar{T}_{,a} \bar{T}_{;bc} \bar{u}^c + 2 (\bar{L} - 2) \bar{h}^{ab} \bar{T}_{,a} \bar{T}_{,b} \bar{T}_{,c} \bar{u}^c = 0 \quad (54)$$

is valid on  $[-c, 0)$ . We also know that  $\bar{\Sigma}^a \bar{T}_{,a}$  is at least  $C^1$ ,  $\bar{h}^{ab} \bar{T}_{,a} \bar{T}_{;bc} \bar{u}^c$  is at least  $C^2$  and  $\bar{h}^{ab} \bar{T}_{,a} \bar{T}_{,b} \bar{T}_{,c} \bar{u}^c$  is at least  $C^3$  on an open neighbourhood of  $\bar{T} = 0$  in  $\bar{\mathcal{M}}$ . We know that as  $\bar{T} \rightarrow 0^-$ ,  $\frac{\bar{\Omega}}{\bar{\Omega}'} \rightarrow 0$  and therefore as  $\bar{T} \rightarrow 0$

$$\left( \frac{\bar{\Omega}}{\bar{\Omega}'} \right)^2 \bar{\Sigma}^a \bar{T}_{,a} + 2 \frac{\bar{\Omega}}{\bar{\Omega}'} \bar{h}^{ab} \bar{T}_{,a} \bar{T}_{;bc} \bar{u}^c \rightarrow 0 \quad (55)$$

$$\implies 2 (\bar{L} - 2) \bar{h}^{ab} \bar{T}_{,a} \bar{T}_{,b} \bar{T}_{,c} \bar{u}^c = 0. \quad (56)$$

Now  $\bar{\lambda} \neq 2$  and  $\bar{u}^c \bar{T}_{,c} \neq 0$  on  $\bar{T} = 0$  so therefore  $\bar{h}^{ab} \bar{T}_{,a} \bar{T}_{,b} = 0$  on  $\bar{T} = 0$  and therefore  $\bar{u}$  is orthogonal to  $\bar{T} = 0$ .  $\square$



**Theorem 10** If the spacetime  $(\mathcal{M}, g)$  satisfies condition 8 and  $1 < \bar{\lambda} < +\infty$  then there exists a limiting  $\gamma$ -law equation of state  $p = (\gamma - 1)\mu$  as the PIU/FIU is approached, where  $\gamma = \frac{2}{3}(2 - \bar{\lambda})$ .

*Proof*

For a perfect fluid source, the Einstein Field Equations reveal the following [12]

$$A = \mu = \bar{\Omega}^{-2} \left[ \left( \frac{\bar{\Omega}'}{\bar{\Omega}} \right)^2 \left[ 3 (\bar{u}^a \bar{T}_{,a})^2 + (1 - 2\bar{L}) \bar{h}^{ab} \bar{T}_{,a} \bar{T}_{,b} \right] - 2 \frac{\bar{\Omega}'}{\bar{\Omega}} \bar{h}^{ab} \bar{T}_{;ab} + \bar{A} \right] \quad (57)$$

$$B = p = \bar{\Omega}^{-2} \left[ \left( \frac{\bar{\Omega}'}{\bar{\Omega}} \right)^2 \left[ (1 - 2\bar{L}) (\bar{u}^a \bar{T}_{,a})^2 + \frac{1}{3} (1 + 4\bar{L}) \bar{h}^{ab} \bar{T}_{,a} \bar{T}_{,b} \right] + \frac{2\bar{\Omega}'}{3\bar{\Omega}} (\bar{h}^{ab} \bar{T}_{;ab} - 3\bar{u}^a \bar{u}^b \bar{T}_{;ab}) + \bar{B} \right] \quad (58)$$

Now we know that as  $\bar{T} \rightarrow 0$ ,  $\bar{\Omega} \rightarrow +\infty$ ,  $\bar{\Omega}'/\bar{\Omega} \rightarrow +\infty$  and the unphysical metric is  $C^2$ , so we can approximate the equations above as

$$\mu \approx 3\bar{M}^2 (\bar{u}^a \bar{T}_{,a})^2 \quad (59)$$

$$p \approx \bar{M}^2 (1 - 2\bar{\lambda}) (\bar{u}^a \bar{T}_{,a})^2 \quad (60)$$

where  $\bar{M} = \bar{\Omega}'/\bar{\Omega}^2$ . The difficulty here, unlike for the case of the IPS/IFS, is that  $\bar{M}$  does not necessarily diverge to  $+\infty$ ; it will have different behaviour for  $\bar{\Omega}(0) = +\infty$  as compared to when  $\bar{\Omega}(0) = 0$ . The behaviour of  $\bar{M}$  has previously been determined [4] and as such we simply exploit the results below.

For  $1 < \bar{\lambda} < 2$ ,  $\lim_{\bar{T} \rightarrow 0^-} \bar{M} \in \mathbb{R}^+ \cup \{0\}$ , while for  $\bar{\lambda} > 2$   $\lim_{\bar{T} \rightarrow 0^-} \bar{M} \in \mathbb{R}^+ \cup \{+\infty\}$ . The case  $\bar{\lambda} = 2$  is eliminated by the definition of a PIU/FIU.

The asymptotic behaviour for  $\mu$  and  $p$  as  $\bar{T} \rightarrow 0^-$  is as follows

$$\mu \approx 3\bar{M}^2 (\bar{u}^a \bar{T}_{,a})^2 \rightarrow \alpha \in \mathbb{R}^+ \cup \{0\} \quad 1 < \bar{\lambda} < 2 \quad (61)$$

$$\mu \approx 3\bar{M}^2 (\bar{u}^a \bar{T}_{,a})^2 \rightarrow \beta \in \mathbb{R}^+ \cup \{+\infty\} \quad \bar{\lambda} > 2 \quad (62)$$

$$p \approx \bar{M}^2 (1 - 2\bar{\lambda}) (\bar{u}^a \bar{T}_{,a})^2 \rightarrow \gamma \in \mathbb{R}^- \cup \{0\} \quad 1 < \bar{\lambda} < 2 \quad (63)$$

$$p \approx \bar{M}^2 (1 - 2\bar{\lambda}) (\bar{u}^a \bar{T}_{,a})^2 \rightarrow \delta \in \mathbb{R}^- \cup \{-\infty\} \quad \bar{\lambda} > 2 \quad (64)$$

It follows that for  $1 < \bar{\lambda} < +\infty$

$$p \approx \frac{1}{3} (1 - 2\bar{\lambda}) \mu \quad (65)$$

and so there exists a limiting  $\gamma$ -law equation of state

$$p = (\gamma - 1) \mu \quad (66)$$

where  $\gamma = \frac{2}{3}(2 - \bar{\lambda})$ .  $\square$

We note that a solution is asymptotic dust if and only if  $\bar{M} \rightarrow 0$  as  $\bar{T} \rightarrow 0^-$ . This can only occur for  $1 < \bar{\lambda} < 2$ .

**Theorem 11** If the spacetime  $(\mathcal{M}, g)$  satisfies condition 8 and  $\bar{\lambda} = +\infty$  then there exists a limiting equation of state  $p \approx -\frac{2}{3}\bar{L}\mu$  as the PIU/FIU is approached.

*Proof*

Again we use the following equations from the Einstein Field Equations

$$A = \mu = \bar{\Omega}^{-2} \left[ \left( \frac{\bar{\Omega}'}{\bar{\Omega}} \right)^2 \left[ 3 (\bar{u}^a \bar{T}_{,a})^2 + (1 - 2\bar{L}) \bar{h}^{ab} \bar{T}_{,a} \bar{T}_{,b} \right] - 2 \frac{\bar{\Omega}'}{\bar{\Omega}} \bar{h}^{ab} \bar{T}_{;ab} + \bar{A} \right] \quad (67)$$

$$B = p = \bar{\Omega}^{-2} \left[ \left( \frac{\bar{\Omega}'}{\bar{\Omega}} \right)^2 \left[ (1 - 2\bar{L}) (\bar{u}^a \bar{T}_{,a})^2 + \frac{1}{3} (1 + 4\bar{L}) \bar{h}^{ab} \bar{T}_{,a} \bar{T}_{,b} \right] + \frac{2}{3} \frac{\bar{\Omega}'}{\bar{\Omega}} (\bar{h}^{ab} \bar{T}_{;ab} - 3 \bar{u}^a \bar{u}^b \bar{T}_{;ab}) + \bar{B} \right] \quad (68)$$

Now we know that as  $\bar{T} \rightarrow 0^-$  both  $\bar{\Omega}$  and  $\bar{\Omega}'/\bar{\Omega}$  diverge and the unphysical metric is  $C^2$ . We also know from equation 56 that  $\bar{h}^{ab} \bar{T}_{,a} \bar{T}_{,b} \rightarrow 0$  as  $\bar{T} \rightarrow 0^-$ . Thus we can approximate the above equations as

$$\mu \approx 3\bar{M}^2 (\bar{u}^a \bar{T}_{,a})^2 \quad (69)$$

$$p \approx -2\bar{L}\bar{M}^2 (\bar{u}^a \bar{T}_{,a})^2 \quad (70)$$

where  $\bar{M} = \bar{\Omega}'/\bar{\Omega}^2$ .

For  $\bar{\lambda} = +\infty$ ,  $\lim_{\bar{T} \rightarrow 0^-} \bar{M}(\text{exists}) = \epsilon \in \mathbb{R}^+ \cup \{+\infty\}$ .

This means that, as  $\bar{T} \rightarrow 0^-$ ,

$$\mu \approx 3\bar{M}^2 (\bar{u}^a \bar{T}_{,a})^2 \rightarrow \zeta \in \mathbb{R}^+ \cup \{+\infty\} \quad (71)$$

$$p \approx -2\bar{L}\bar{M}^2 (\bar{u}^a \bar{T}_{,a})^2 \rightarrow -\infty \quad (72)$$

The limiting equation of state for  $\bar{\lambda} = +\infty$  is therefore

$$p \approx -\frac{2}{3}\bar{L}\mu \quad \square \quad (73)$$

The following lemma is the equivalent of lemma 6.1.2 from [12] but for a PIU/FIU.

**Lemma 12** Define a function  $G(t)$  on  $(0, t_1]$  by

$$G(t) \equiv \int_t^{t_1} a(u) du \quad (74)$$

An FRW model satisfies all but one of the conditions of a PIU/FIU if  $G(t) \rightarrow \eta \in \mathbb{R}^+$  as  $t \rightarrow 0$ .

*Proof*

The only condition that this does not immediately satisfy is the condition concerning  $\bar{\lambda}$ .

We can see that the function  $G(t)$  is a positive and strictly monotonically decreasing function on  $(0, t_1]$  decreasing to zero at  $t_1$ . This guarantees that  $\lim_{T \rightarrow 0} a(t)$  always exists, so that the condition  $a(t) \rightarrow \eta \in \mathbb{R}^+$  as  $t \rightarrow 0$  only eliminates those FRW models for which  $G(t) \rightarrow +\infty$  as  $t \rightarrow 0$ .

If we assume that  $G(t) \rightarrow \eta \in \mathbb{R}^+$  as  $t \rightarrow 0$ , define  $T$  on  $(0, t_1]$  by

$$T(t) \equiv \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^t \frac{1}{a(u)} du \quad (75)$$

We see that  $T(t)$  is well defined and positive, strictly monotonically increasing on  $(0, t_1]$  and that  $T(t) \rightarrow 0$  as  $t \rightarrow 0$ .

The rest of the proof is essentially identical to that seen in Ericksson [12].  $\square$

**Corollary 13** If an FRW model satisfies the conditions of Lemma 12 then the fluid flow is regular at the PIU/FIU.

*Proof*

The proof is the same as in Ericksson [12] and as such we direct the reader there.  $\square$

If the spacetime admits a PIU/FIU then Höhn and Scott [4] has shown that  $\bar{\lambda} > 1$  and  $\bar{\lambda} \neq 1, 2$ ; this fact will be key to determining the behaviour of a PIU/FIU.

For convenience, we now concisely present various properties of the PIU/FIU. These results are due to [4] or the previous section.

For a PIU/FIU structure

$$\begin{aligned}
 \bar{\lambda} &= \lim_{\bar{T} \rightarrow 0} \bar{L} = \lim_{\bar{T} \rightarrow 0} \frac{\bar{\Omega}'' \bar{\Omega}}{\bar{\Omega}'^2} \\
 \bar{M} &= \frac{\bar{\Omega}'}{\bar{\Omega}^2} \\
 a &\equiv \bar{\Omega} \\
 p &\approx \bar{M}^2 (1 - 2\bar{\lambda}) (\bar{u}^a \bar{T}_{,a})^2 \\
 \mu &\approx 3\bar{M}^2 (\bar{u}^a \bar{T}_{,a})^2
 \end{aligned} \tag{76}$$

In the limit that  $\bar{T} \rightarrow 0$ , the variables in equation 76 adopt the following asymptotic form

$\bar{\lambda}$	$\bar{M}$	$a$	$p$	$\mu$
$1 < \bar{\lambda} < 2$	$\beta \in \mathbb{R}^+ \cup \{0\}$	$+\infty$	$\gamma \in \mathbb{R}^- \cup \{0\}$	$\delta \in \mathbb{R}^+ \cup \{0\}$
$2 < \bar{\lambda} < +\infty$	$\alpha \in \mathbb{R}^+ \cup \{+\infty\}$	$+\infty$	$\epsilon \in \mathbb{R}^- \cup \{-\infty\}$	$\phi \in \mathbb{R}^+ \cup \{+\infty\}$
$\bar{\lambda} = +\infty$	$\alpha \in \mathbb{R}^+ \cup \{+\infty\}$	$+\infty$	$-\infty$	$\phi \in \mathbb{R}^+ \cup \{+\infty\}$

Using all our preliminary results, we now present the following theorem that provides the final piece in puzzle for the analysis of the FRW solution case  $\eta_0 \geq 1$ .

**Theorem 14** (The FRW PIU/FIU Result) An FRW model admits a PIU/FIU with  $1 < \bar{\lambda} < +\infty$  at which the fluid flow is regular if and only if  $\lim_{T \rightarrow 0} \frac{\ddot{a}a}{\dot{a}^2}$  exists and is greater than zero and  $\lim_{t \rightarrow 0} a(t) \rightarrow +\infty$

*Proof*

Assuming the existence of a PIU/FIU first, we seek to prove  $\lim_{T \rightarrow 0} \frac{\ddot{a}a}{\dot{a}^2}$ .

The equation of state is

$$p = (\gamma - 1) \mu [1 + f(t)] \quad 1 < \lambda < +\infty \quad (77)$$

$$(78)$$

where  $\gamma = \frac{2}{3}(2 - \lambda)$  and  $f(t) = o(1)$ . We see that if we use the conservation equations for a perfect fluid source [6] we get

$$\dot{\mu} = -3(\mu + p) \frac{\dot{a}}{a} \quad (79)$$

$$= -3(\mu + (\gamma - 1)\mu[1 + o(1)]) \frac{\dot{a}}{a} \quad (80)$$

$$= -3\gamma\mu \frac{\dot{a}}{a} [1 + o(1)] \quad (81)$$

which means we can write the same as Ericksson [12]

$$\frac{\dot{\mu}}{\mu} = -3\gamma[1 + o(1)] \frac{\dot{a}}{a} \quad (82)$$

$$\ln \mu|_t^{t_1} = -3\gamma[1 + o(1)] \ln a|_t^{t_1} \quad (83)$$

$$\ln \mu = -3\gamma[1 + o(1)] \ln a \quad (84)$$

$$\mu a^2 = a^{2-3\gamma} \quad (85)$$

We need to examine what happens with  $\gamma = \frac{2}{3}(2 - \lambda)$  as we vary  $\bar{\lambda}$

$$1 < \bar{\lambda} < 2 \implies 0 < \gamma < \frac{2}{3} \quad (86)$$

$$\bar{\lambda} = 2 \implies \gamma = 0 \quad (87)$$

$$\bar{\lambda} > 2 \implies \gamma < 0 \quad (88)$$

So it means that we can say, because  $a \rightarrow +\infty$  as  $T \rightarrow 0$  for a PIU/FIU

$$\lim_{\bar{T} \rightarrow 0} \mu a^2 \rightarrow +\infty \quad (89)$$

This even includes the finite values that  $\mu$  may take because  $\mu$  is nonnegative.

This means that due to the Einstein field equation,

$$3\dot{a}^2 = \mu a^2 - 3K \quad (90)$$

it means  $\lim_{\bar{T} \rightarrow 0} \dot{a}^2 \rightarrow +\infty$ .

Now we can use other field equations and evaluate  $\ddot{a}/a$ .

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\mu + 3p) \quad (91)$$

$$= -\frac{1}{6}(\mu + 3(\gamma - 1)\mu[1 + o(1)]) \quad (92)$$

$$= -\frac{\mu}{6}(3\gamma - 2 + o(1)) \quad (93)$$

Now we can multiply this by  $\frac{a^2}{\dot{a}^2}$

$$\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\mu a^2}{6\dot{a}^2}(3\gamma - 2)[1 + o(1)] \quad (94)$$

$$= -\frac{1}{6}(3\gamma - 2)\left(3 + \frac{3K}{\dot{a}^2}\right)[1 + o(1)] \quad (95)$$

As  $\dot{a}^2 \rightarrow +\infty$ , it means

$$\frac{\ddot{a}a}{\dot{a}^2} \rightarrow -\frac{1}{2}(3\gamma - 2) \quad (96)$$

For a PIU/FIU  $\gamma < \frac{2}{3}$  and as such this value is always going to be positive. Hence for a PIU/FIU  $\frac{\ddot{a}a}{\dot{a}^2} > 0$ .

We will now prove the converse, again following the lead of [12]. Assume that the FRW spacetime has  $\lim_{\bar{T} \rightarrow 0} \frac{\ddot{a}a}{\dot{a}^2} > 0$  exists. Let the limit of  $\frac{\ddot{a}a}{\dot{a}^2}$  be written as  $\bar{\lambda} - 1$ . As  $\bar{\lambda} > 1$  means this value will always be positive.

We have assumed  $\lim_{t \rightarrow 0} \frac{\ddot{a}a}{\dot{a}^2} = \bar{\lambda} - 1 > 0$  and as such

$$a = (\bar{\lambda} - 1) \frac{\dot{a}^2}{\ddot{a}} [1 + o(1)] \quad (97)$$

$$\int_t^{t_1} a(u) du = (\bar{\lambda} - 1) \left( \int_t^{t_0} \frac{\dot{a}^2}{\ddot{a}} du \right) [1 + o(1)] \quad (98)$$

We need to find out if  $\int_t^{t_0} \frac{\dot{a}^2}{a} du$  is positive because if it is then we know that  $\int_t^{t_1} a(u) du$  will be positive and we will have three of four of the conditions for a PIU/FIU satisfied.

We know that  $\dot{a}^2 \rightarrow +\infty$  which means  $\ln \dot{a}^2 \rightarrow +\infty = 2 \ln \dot{a}$  as such there will be an interval,  $(0, t_0]$  such that  $\dot{a} > 0$ . If we rewrite this as

$$2 \ln \dot{a} = 2 \int \frac{\ddot{a}}{\dot{a}} du \rightarrow +\infty \quad (99)$$

So we know that because  $\dot{a} > 0$  it will mean that  $\ddot{a} > 0$ . If it was less than zero then the integral would not diverge to positive infinity, there would be a negative sign appearing.

This helps us because it means that  $\int_t^{t_0} \frac{\dot{a}^2}{a} du$  is positive and as such  $\int_t^{t_1} a(u) du > 0$  which means that we have three out of four conditions satisfied for a PIU/FIU. We need to check  $\bar{L}$ . Set

$$\bar{L}(\bar{T}) = \frac{\bar{\Omega}'' \bar{\Omega}}{\bar{\Omega}^2} \quad (100)$$

$$= \frac{a(t) (\ddot{a}(t) a^2(t) + \dot{a}^2(t) a(t))}{\dot{a}^2(t) a^2(t)} \quad (101)$$

$$= \frac{\ddot{a} a}{\dot{a}^2} + 1 \quad (102)$$

But because  $\frac{\ddot{a} a}{\dot{a}^2} = \bar{\lambda} - 1$  as  $t \rightarrow 0$  we see  $\bar{L} \rightarrow \bar{\lambda}$  as  $t \rightarrow 0$ .

This proves the theorem.  $\square$

Previously we only considered a PIU/FIU with a finite value of  $\bar{\lambda}$  but now we will examine what happens for  $\bar{\lambda} = +\infty$ .

**Theorem 15** An FRW model admits a PIU/FIU with  $\bar{\lambda} = +\infty$  at which the fluid flow is regular if and only if  $\lim_{\bar{T} \rightarrow 0} \frac{\ddot{a} a}{\dot{a}^2} = +\infty$  and  $\lim_{t \rightarrow 0} a(t) \rightarrow +\infty$ .

*Proof*

We previously showed, similar to Ericksson [12], that if  $\bar{\lambda} = +\infty$  then  $\mu \in$

$\mathbb{R}^+ \cup \{+\infty\}$  and

$$p = -\frac{2}{3}\bar{L}\mu[1 + o(1)]. \quad (103)$$

Putting this into the following

$$\dot{\mu} = -3(\mu + p)\frac{\dot{a}}{a} \quad (104)$$

$$= 2\bar{L}\mu[1 + o(1)]\frac{\dot{a}}{a} \quad (105)$$

So it means that

$$\frac{\dot{\mu}}{\mu} = 2\bar{L}[1 + o(1)]\frac{\dot{a}}{a} \quad (106)$$

$$\therefore 2\frac{\dot{a}}{a} = o\left(\frac{\dot{\mu}}{\mu}\right) \quad (107)$$

$$\implies 2 \int_t^{t_1} \frac{\dot{a}(u)}{a(u)} du = o\left(\int_t^{t_1} \frac{\dot{\mu}(u)}{\mu(u)} du\right) \quad (108)$$

$$\implies 2(\ln a(t_1) - \ln a(t)) = o(\ln \mu(t_1) - \ln \mu(t)) \quad (109)$$

$$-2\ln a(t) = o(\ln \mu(t)) \quad (110)$$

$$a^{-2} = o(\mu) \quad (111)$$

$$a^{-2}\mu^{-1} = o(1) \quad (112)$$

and hence  $a^2\mu \rightarrow +\infty$ , which again means that  $\dot{a}^2 \rightarrow +\infty$ . Just to be clear, we know that this is true even if  $\mu \in \mathbb{R}^+$  because  $a \rightarrow +\infty$ . We now demonstrate the following

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\mu + 3p) \quad (113)$$

$$= -\frac{1}{6}(\mu - 2\bar{L}\mu[1 + o(1)]) \quad (114)$$

$$= \frac{1}{3}\bar{L}\mu[1 + o(1)] \quad (115)$$

$$\implies \frac{\ddot{a}a}{\dot{a}^2} = \frac{1}{3}\bar{L}\mu\frac{a^2}{\dot{a}^2}[1 + o(1)] \quad (116)$$

$$= \frac{1}{3}\bar{L}\frac{3\dot{a}^2 + 3K}{\dot{a}^2}[1 + o(1)] \quad (117)$$

$$= \bar{L}\left(1 + \frac{K}{\dot{a}^2}\right)[1 + o(1)] \quad (118)$$

and because  $\dot{a}^2 \rightarrow +\infty$  it is obvious that  $\frac{\ddot{a}a}{\dot{a}^2}$  diverges to positive infinity as  $\bar{T} \rightarrow 0$  because we have assumed  $\bar{\lambda} \rightarrow +\infty$ .

We now want to prove the converse. Consider an FRW model for which  $\frac{\ddot{a}a}{\dot{a}^2} \rightarrow +\infty$  as  $t \rightarrow 0$ . We know that  $a(t) > 0$  and  $\dot{a}^2 > 0$  on a neighbourhood  $(0, t_1]$  so this means



that there must exist a  $t_2 \in (0, t_1]$  such that  $\ddot{a}(t) > 0$  on  $(0, t_2]$ . So what this means is that  $\dot{a}(t)$  must be a strictly monotonically increasing function on  $(0, t_2]$  and hence as  $t \rightarrow 0$ ,  $\dot{a} \rightarrow \alpha \in \mathbb{R}^+$  or  $\dot{a} \rightarrow +\infty$ .

Let us take

$$\frac{\ddot{a}a}{\dot{a}^2} = F(t) \quad \text{where } \implies F(t) \rightarrow \infty \quad \text{as } t \rightarrow 0 \quad (119)$$

$$\implies \frac{\dot{a}^2}{\ddot{a}a} = o(1) \quad (120)$$

So we know that

$$\frac{1}{a} = o\left(\frac{\ddot{a}}{\dot{a}^2}\right). \quad (121)$$

Now consider

$$\int_t^{t_2} \frac{\ddot{a}(u)}{\dot{a}^2(u)} du = \frac{1}{\dot{a}(t_2)} - \frac{1}{\dot{a}(t)} \quad (122)$$

as we know that  $\dot{a} \rightarrow \alpha \in \mathbb{R}^+ \cup \{+\infty\}$  it is true that

$$\int_t^{t_2} \frac{\ddot{a}(u)}{\dot{a}^2(u)} du \rightarrow \gamma \in \mathbb{R}^+ \quad (123)$$

and hence that

$$\int_t^{t_1} \frac{1}{a(u)} du \rightarrow \chi \in \mathbb{R}^+ \quad (124)$$

So we know that, because of Lemma 12 and Corollary 13, the FRW model will satisfy three conditions for a PIU/FIU and we will prove the last condition now. Set

$$\bar{L}(\bar{T}) = \frac{\bar{\Omega}''\bar{\Omega}}{\bar{\Omega}'^2} \quad (125)$$

$$= \frac{a(t) (\ddot{a}(t)a^2(t) + \dot{a}^2(t)a(t))}{\dot{a}^2(t)a^2(t)} \quad (126)$$

$$= \frac{\ddot{a}a}{\dot{a}^2} + 1 \quad (127)$$

But because  $\frac{\ddot{a}a}{\dot{a}^2} = +\infty$  as  $t \rightarrow 0$  we can say  $\bar{L} \rightarrow \bar{\lambda} = +\infty$  as  $t \rightarrow 0$ .

We have been able to prove the theorem.  $\square$

So what this theorem does, in conjunction with The FRW Result [12], is prove that the FRW solution studied in this paper (with  $\eta_0 \geq 1$ ) cannot admit an IPS/IFS or a

PIU/FIU. It cannot admit an IPS/IFS because  $\lim_{T \rightarrow 0} \frac{\ddot{a}a}{a^2} \geq 0$  [12] and it cannot admit a PIU/FIU because  $a(t)$  does not diverge as  $t \rightarrow 0^\pm$ .

#### 4. Conclusions and Further Outlook

In this paper we have shown that there exist large classes of FRW models that obey conformal definitions from the Quiescent Cosmology framework. We were able to show that FRW models, whose scale factor is represented as a generalised and finite series expansion, will represent an FIU if that scale factor diverges (for  $\eta_0 < 0$ ) at a particular instant in time. When the scale factor vanishes, we showed that some models represent an IPS/IFS (for  $\eta_0 \in (0, 1)$ ) and were able to prove that the remaining cases (for  $\eta_0 \geq 1$ ) cannot represent an IPS/IFS or a PIU/FIU.

The FRW Result of Ericksson [12] provides a simple method of determining when an FRW solution can admit an IPS. In this paper, we formed an equivalent statement for the FIU. In proving the FRW Result, Ericksson also determined a gamma-law equation of state for FRW models that represent an IPS. In the course of providing relevant background information for The FRW (PIU/FIU) Result, we were also able to demonstrate that there exists a gamma-law equation of state for FRW solutions that admit an FIU. With the two FRW results (for the IPS and FIU respectively), we now know it will always be clear when an FRW model admits one of the isotropic structures from Quiescent Cosmology.

The synthesis between the FRW and Quiescent Cosmology frameworks is a natural one due to their respective ability at describing the observable Universe. Quiescent Cosmology is attempting to describe the very early Universe and hence the Big Bang while FRW models offer reasonable explanations for both the Big Bang and the current large scale state of the Universe. By studying both approaches to cosmology, it may be possible to reveal even more information about the Big Bang and the current Universe.

It is important to further our knowledge of FRW models that cannot obey either

of the isotropic definitions from Quiescent Cosmology as they could provide insight for possible future alternate isotropic structures. It is also important to understand the link between the IPS/IFS and FIU, and the Big Bang/Crunch and Big Rip respectively - this can be achieved by even more study of the overlap between the FRW and Quiescent Cosmology frameworks.

## References

- [1] L. Fernández Jambrina and R. Lazkoz. Geodesic Behaviour around Cosmological Milestones. *Journal of Physics*, 66:012015, 2007.
- [2] C. Cattoën and M. Visser. Necessary and Sufficient Conditions for Big Bangs, Bounces, Crunches, Rips, Sudden Singularities and Extremality Events. *Classical and Quantum Gravity*, 22:4913–4930, 2005.
- [3] J.D. Barrow. Quiescent Cosmology. *Nature*, 272:211–215, 1978.
- [4] P. A. Höhn and S. M. Scott. Encoding Cosmological Futures with Conformal Structures. *Classical and Quantum Gravity*, 26:035019, 2009.
- [5] S. W. Goode and J. Wainwright. Isotropic Singularities in Cosmological Models. *Classical and Quantum Gravity*, 2:99–115, 1985.
- [6] G. F. R. Ellis. Relativistic Cosmology. In *General Relativity and Cosmology; Proceedings of the International School of Physics - Enrico Fermi, Varenna, Italy; United States; 30 June-12 July 1969*, pages 104–182, 1971.
- [7] R. R. Caldwell. A Phantom Menace. *Physics Letters B*, 545:23–29, 2002.
- [8] R. R. Caldwell, M. Kamionkowski, and N. N. Weinberg. Phantom Energy: Dark Energy with  $w < -1$  Causes a Cosmic Doomsday. *Physical Review Letters*, 91(071301):071301, 2003.
- [9] J.D. Barrow. More General Sudden Singularities. *Classical and Quantum Gravity*, 21:5619–5622, 2004.
- [10] C. W. Misner. Transport Processes in the Primordial Fireball. *Nature*, 214:40–41, 1967.
- [11] S.M. Scott. The General Vorticity Result. (In preparation).
- [12] G. Erickson. *The Isotropic Singularity in Cosmological Models*. PhD thesis, The Australian National University, 2003.